

Gaugino Determinant in Supersymmetric Yang-Mills Theory

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Abstract

We resolve an ambiguity in the sign of the determinant of a single Weyl fermion, such as the gaugino in supersymmetric models. Positivity of this determinant is necessary for application of QCD inequalities and lattice Monte Carlo methods to supersymmetric Yang-Mills models.

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In this note we address an ambiguity in the sign of the gaugino determinant. This ambiguity is important, in that it affects the possible non-perturbative methods available to study supersymmetric Yang-Mills [1] (SYM) theory. In particular, the application of QCD inequalities [2] as well as lattice Monte Carlo methods [3, 4] to SYM rely on the positivity of the fermion measure. Because SYM is vector like (one can write a gauge invariant majorana mass for the gaugino), one is tempted to conclude immediately that the measure is positive definite. However, the definition of the gaugino determinant is somewhat subtle, and some technical machinery (an index theorem in five dimensions) is necessary to resolve the sign ambiguity.

The determinant for a single Weyl fermion cannot be straightforwardly defined because the Weyl operator maps the vector space of left-handed spinors into the space of right-handed spinors, and therefore fails to define an eigenvalue problem [5]. Rather, one usually defines the determinant in terms of the eigenvalues of the Dirac operator [6]. Naively, one simply writes

$$\det i \not{D}_{\text{Weyl}} = (\det i \not{D}_{\text{Dirac}})^{1/2} . \quad (1)$$

However, this definition leads to a sign ambiguity, as first noticed by Witten [6]. Suppose we define the Weyl determinant for some fiducial background gauge configuration (which we take here to be $A_\mu^0(x) = 0$) as the product of only the *positive* eigenvalues of $i \not{D}_{\text{Dirac}}$. Once this choice is made, there is no additional freedom, and the Weyl determinant is defined for all $A_\mu(x)$ by the condition that it vary smoothly as the gauge field is varied (see [7] for further discussion). This condition requires that we continue to take the Weyl determinant as the product of the *same* eigenvalues, which flow continuously as the gauge field is varied. The sign ambiguity arises because some of the originally positive eigenvalues can become negative for some background fields. If an odd number do so, the determinant becomes negative and therefore spoils the positivity property of the measure. This is problematic for Monte Carlo simulations, because the functional integral loses its statistical interpretation.

Witten investigated these possible sign changes in the case of $SU(2)$ with a Weyl fermion in the fundamental representation. There, the model is intrinsically inconsistent, because the Weyl determinant changes sign under certain topologically non-trivial gauge transformations. In SYM we are interested in a related behavior which, while not rendering the theory inconsistent, would make it more difficult to study at the non-perturbative level.

Fortunately, one can show that for a Weyl fermion in the adjoint representation, the eigenvalue flow always involves an even number of eigenvalues crossing zero. Hence the sign of the gaugino determinant is constant and can be chosen to be positive. We use the machinery of reference [6]. There, it is demonstrated that the flow of eigenvalues of the four dimensional Dirac operator can be related to the number of zero modes of the five dimensional

Dirac operator \mathcal{D}_5 on a cylinder $S^4 \times R$, consisting of the smooth interpolation of the fiducial gauge field to the gauge field of interest, $A_\mu(x)$. The mod two Atiyah-Singer index theorem [8] gives the number of zero modes of \mathcal{D}_5 modulo 2 as twice $C(R)$ (the Casimir of the fermion representation) times an integer topological invariant related to $\pi_4(G)$, where G is the gauge group. In [6] the index theorem is applied to cases in which the (four dimensional) gauge field of interest is a gauge transform of the fiducial gauge field ($A_\mu^0(x) = 0$):

$$A_\mu(x) = iU^\dagger \partial_\mu U(x) \quad . \quad (2)$$

We do not wish to restrict ourselves to this case, as we need the sign of the Weyl determinant for arbitrary $A_\mu(x)$. In order to consider arbitrary gauge fields, we will exploit the fact that the number of zero modes of \mathcal{D}_5 is conserved mod 2 under any smooth deformation of the five dimensional gauge configuration. This result is easy to see since \mathcal{D}_5 is a real, antisymmetric operator whose non-zero eigenvalues are purely imaginary and occur in pairs. Any flow of these eigenvalues under the smooth deformation of the five dimensional gauge field will change the number of zero modes by a multiple of two, leaving the sign of the determinant intact. Now note that if two four dimensional gauge configurations are smooth deformations of each other, the five dimensional configuration consisting of the interpolation between the two is itself smoothly deformable to a five dimensional configuration which is just the constant (in x_5) interpolation one obtains by extending one of the four dimensional configurations into the fifth dimension. The latter will of course exhibit no level crossing, so the former must exhibit only an even number of crossings. Thus the signs of the determinants of two four dimensional gauge configurations must be the same if they are smoothly related.

Using the above result, the sign of the determinant for arbitrary (four dimensional) $A_\mu(x)$ can be determined by using the index theorem on a suitable vacuum configuration (2) which is smoothly connected to $A_\mu(x)$. Due to the factor of $2C(R)$ from the index theorem, an integer-valued Casimir then guarantees that the eigenvalues of $\mathcal{D}_{\text{Dirac}}$ used to define the Weyl determinant only cross zero in even multiples, preserving the sign of the Weyl determinant.

It remains to examine whether an arbitrary (four dimensional) $A_\mu(x)$ can be smoothly connected to some “nearest” vacuum configuration (2). In order to do this, we first carefully examine the boundary conditions imposed on our field configurations. In order to obtain a π_4 classification of field configurations in five dimensions, we require that the field approach a pure gauge on the surface at infinity:

$$A_\mu(|x| \rightarrow \infty) \rightarrow iU^\dagger \partial_\mu U(x) \quad . \quad (3)$$

The functions $U(x)$ map $S^4 \rightarrow G$ and are classified by $\pi_4(G)$, allowing the application of the index theorem to \mathcal{D}_5 . If we consider the five dimensional space to be $R^4 \times R$, then

each slice at fixed x_5 obeys boundary conditions like (3), but applied to the surface of R^4 . This allows a classification of each four dimensional configuration by $\pi_3(G)$. First consider the zero winding number sector. Here the vacuum is given by some gauge function $U(x)$ which is smooth on all of R^4 . By a smooth gauge transformation we can take $U(x) = U_0$ constant everywhere, so the vacuum is simply $A_\mu = 0$. The boundary condition on a field configuration in this gauge is that $A_\mu(|x| \rightarrow \infty) = 0$. A smooth interpolation which relates A_μ to the vacuum is simply $A_\mu^t = tA_\mu$.

In the sectors with non-zero winding number under $\pi_3(G)$, (ie non-zero instanton number), there is a potential problem because extending the gauge function $U(x)$ from the surface at infinity (S^3) into the interior of R^4 cannot be done without a singularity. This means that the “nearest” vacuum to a topologically non-trivial configuration is itself singular, and there is the possibility that something discontinuous can happen during the interpolation. However, this possibility can be excluded because the interpolation from A_μ to the vacuum is related by a large gauge transform to an interpolation in the zero-winding sector which is smooth. Since the eigenvalues themselves are gauge invariant, the interpolations in all sectors are smooth.

Actually, the zero-winding sector of configurations is sufficient to deduce the properties of a model with zero theta angle. (θ must be zero in any case to preserve positivity of the functional measure.) This is because, in the infinite volume limit, the only remnant of the boundary conditions placed on the system is $\theta \int F\tilde{F}$. When investigating the $\theta = 0$ theory, we are therefore allowed to take any boundary conditions. In particular, we can define the theory in the zero-winding sector without changing any of the physics.

The fourth homotopy group $\pi_4(G)$ is non-zero for $SU(2)$, $O(N < 6)$ and $Sp(N)$ (any N). These are the only cases in which the ambiguity can arise (although this is far from clear *a priori*!). For $SU(2)$, the case most likely to be of interest in lattice simulations [4], $\pi_4(SU(2)) = \mathbf{Z}_2$. In this case the Casimir of the adjoint representation is 2 ($C_{\text{adj}}(SU(N)) = N$), so the sign of the Weyl determinant never fluctuates. In the other cases, we have $C_{\text{adj}}(SO(2N+1)) = 4N - 2$, $C_{\text{adj}}(SO(2N)) = 4N - 4$ and $C_{\text{adj}}(Sp(N)) = N + 1$, so the determinants in these theories behave similarly.

We conclude by noting that our analysis is also of use in the study of chiral gauge theories. In some recent proposals for the lattice realizations of such theories [7, 9], the determinant is again constructed from the product of half of the Dirac eigenvalues, with additional phase information residing in functional Jacobian factors that result from fermionic integration. Our analysis can be used to determine whether there are sign fluctuations beyond those coming from the Jacobian factors. We see that unless the model is afflicted with a global anomaly (and hence inconsistent) there are no such fluctuations.

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